Int. J. Solids Structures Vol. 28, No. 1, pp. 87–104, 1991
Printed in Great Britain.
0020-7683/91 $3.00 + .00

OPTIMUM POINTWISE REINFORCEMENT REQUIREMENTS IN PLASTIC FRACTURING CONTINUA

G. BAKER
Department of Civil Engineering, University of Queensland, St Lucia, Queensland 4067, Australia

(Received 19 December 1988; in revised form 2 September 1990)

Abstract—The non-linear program which governs the provision of optimum pointwise reinforcement in a plastic fracturing continuum under a predetermined stress field is formulated and solved. Tensors which represent excess pointwise capacity are developed for the two- and three-dimensional cases, including skew reinforcement, and it is from these that capacity is minimized. Basic tensor invariants are combined with the Kuhn–Tucker optimality conditions to yield the governing equations, which in some cases can be solved directly. The formulation can be employed with positive, negative or mixed stress fields.

1. INTRODUCTION

The problem of optimally reinforcing a material to withstand an applied stress field has existed for some time. One of the first civil engineering studies in this category was that carried out by Wood (1968), who sought to determine the optimum reinforcement in a slab for a predetermined moment field. He examined the variation in the difference between plastic resistance moment and applied moments on a plane, as the plane was rotated about the point. By requiring a safe solution, Wood minimized the excess with respect to the angle of rotation, and obtained formulae for the required plastic resistance moments. Armer (1968) extended this work to include skew reinforcement.

The process involved is attractive since it yields a simple lower bound solution to the plastic analysis of reinforced concrete slabs, which takes account both of torsion moments (unlike strip methods) and of reinforcement economy. Unfortunately, when applied to mixed moment fields, the procedure reduces to a trial-and-error process, which tends to obscure the principles underlying the solution. Morley (1969) studied the same problem, including skew reinforcement, and produced a series of design charts and curves.

The final result is clearly the same as that derived by Wood and Armer, though the procedure is case dependent even for orthogonal steel. Clark (1976) subsequently extended this work to include in-plane stress as well as applied moments.

Since then, the general field of structural optimization has advanced enormously, particularly for framed structures, in part through the application of the Prager–Shield optimality criteria. Rozvany and Wang (1984), in particular, have extended Prager's work to the analysis of arch grids and cable networks. It is the fact that this work concerns the optimum member layout, and not cross-section size, which is of interest here. Strang and Kohn (1983), for example, were concerned with shape optimization of continua with stresses carried by optimal Michell trusses, rather than optimal reinforcement against failure. Prager structures, unlike Michell frames, have member forces of the same sign throughout, optimizing the location of external forces, and forming globally optimal surface structures. The two forms coincide in certain problems, if the Michell truss is optimized with respect to the load position, though Michell frames generally consist of truss-like continua.

At this stage in their development, however, Prager structures are not relevant to the problem of reinforcing a continuum, since bond renders the constant sign condition unrealistic. Specifically, for reinforced concrete slabs, predetermined resistance moments are employed so that the problem becomes one of shape optimization. Although belonging to an attractive and consistent theory, the extension to complete freedom of reinforcement synthesis appears quite complex. The alternative offered here, which formulates the problems on strength criteria, appears promising.

The purpose of this article is to formulate and solve the non-linear program governing the pointwise provision of optimal reinforcement in a plastic fracturing continuum under a predetermined stress field. It extends previous work to include general three-dimensional problems in reinforced concrete. However, the motivation for, and much of the value in, this work lies in the presentation of a more consistent and generally applicable formulation of the optimization problem. The method of Wood and Armer, though giving the same results for 2D problems as that used here, would become far too complicated if generalized to three dimensions under arbitrary stress fields.

Here, some basic eigenrelations are employed in the problem formulation which, together with the Kuhn–Tucker optimality criteria, provide a consistent approach which embraces the Wood–Armer solution for slabs. The intention is to provide the minimum volume of reinforcement, at a point, to resist the applied stress in excess of the concrete yield surface, at any orientation to the reference axes. The reinforcement provides a varying excess capacity on different planes. For safety and economy, the minimum and maximum values of excess capacity—that is, the eigenvalues of the excess stress tensor—must be examined. The key is quite simply to recognize that the eigenvalues of the excess stress tensor on any plane, $\boldsymbol{\sigma}_\theta$, are the same as those of the tensor in the reference axes, $\boldsymbol{\sigma}$, since the rotation matrix, $\mathbf{R}$, is orthogonal.

In defining material yield and/or failure surfaces, a distinction needs to be drawn between the different failure assumptions, particularly with fracture. It is assumed later, as is sometimes the case in numerical studies of concrete, that the material is plastic fracturing. That is, the material is assumed to retain a constant tensile strength after cracking, effectively thus yielding in tension. Of course, this assumption can be seen as a generalized ultimate limit state view of a tension stiffening model, where a permanent residual stress is assumed across a crack.

If the material is taken to be brittle, then the excess stress will behave like a step function throughout the minimization process, as the material cracks and "uncracks". The solution process is likely to be unstable. Thus, it is better to assume a "no-tension" material if the assumption of plastic fracture is unacceptable. From the generalized view, this requires simply a zero residual at ultimate. For compression, the concrete will be taken as elasto-plastic. Again, to avoid instabilities during the solution process, the biaxial failure surface will be taken to be rectangular, and the multiaxial surfaces as cuboid, rather than stress dependent.

Of course, the reinforcement stiffness affects the stress field which is used for the optimum design. It is not the intention to solve that problem here—the extensions of the solutions presented, coupled with the global formulation for the optimal fibre layout, form part of a continuing study. However, the final section describes, briefly, a simplified finite element synthesis based on the pointwise optimization techniques, but which avoids the need for a full sensitivity analysis.

2. FORMULATION OF THE 3D PROBLEM

The general problem considered is the pointwise determination of the optimal reinforcement requirements for a 3D plastic fracturing continuum under a given stress field. Let the stress tensor of applied stress at a point x in the reference orthogonal axes x, y, z be

$$\mathbf{S}(\mathbf{x}) = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{xy} & S_{yy} & S_{yz} \\ S_{xz} & S_{yz} & S_{zz} \end{bmatrix} \quad \mathbf{x}^{\mathrm{T}} = \langle x, y, z \rangle. \tag{1}$$

Define the standard 3D transformation matrix $\mathbf{R}$, evaluated at the principal angles of the field $\mathbf{S}$, as $\mathbf{Q}$. Then the matrix of eigenvalues, μ_i, of $\mathbf{S}$ (principal stresses) becomes

$$\Lambda(\mathbf{S}) = \mathbf{QSQ}^{\mathrm{T}} = \mathrm{diag}\,(\mu_i). \tag{2}$$

Furthermore, since $\mathbf{Q}$ is orthogonal (as is $\mathbf{R}$), and since expression (2) represents a similarity transform, then the columns $\mathbf{q}_i$, of $\mathbf{Q}$, are the eigenvectors of S (principal directions).

The characterization of the applied/predetermined stress field as positive, negative or mixed can also be related to basic tensor/matrix properties. For a field to be positive, the stresses on any plane must be positive or zero. Thus, in a positive field

$$\det(\mathbf{S}) \geqslant 0; \tag{3a}$$

for positive definiteness $\det(\mathbf{S}) > 0$. In a negative field, where all direct stresses, and hence principal stresses, are negative or zero, the sign of det (S) depends on the dimension of the problem. For this 3D case

$$\det(\mathbf{S}) \leqslant 0. \tag{3b}$$

However, in the 2D problem det (S) must be positive or zero in a negative field; the zero is excluded if negative definite. In a mixed field, the principal stress changes sign. Hence in a 2D problem, $\det(\mathbf{S}) < 0$, but in this 3D case, the sign of det (S) is indeterminate *a priori*.

The material will be characterized as an elastic plastic fracturing material which requires "smeared" perfectly bonded reinforcement to carry all stresses beyond the yield surface. The failure criterion is thus a simple strength criterion along the principal vectors. It is possible to admit a general stress field and yield surface, but the formulation is greatly simplified if positive (tensile), negative (compressive) and mixed fields (tension–tension–compression and tension–compression–compression) are considered separately.

Define the stress tensor corresponding to the failure surface at a point as $\phi(\mathbf{S})$. Here, a simple cuboid surface is chosen so that

$$\boldsymbol{\phi}(\mathbf{S}) = \boldsymbol{\phi} = \mathrm{diag}\,(\phi_i). \tag{4}$$

Moreover, since the stress states are to be considered separately, the failure tensor can be divided into its positive and negative definite components $\boldsymbol{\phi}_+$ and $\boldsymbol{\phi}_-$, respectively. Finally, assuming the material to be isotropic with respect to its ultimate strength, the failure tensors reduce to

$$\begin{aligned} \boldsymbol{\phi}_+ &= f_+\mathbf{I} \\ \boldsymbol{\phi}_- &= f_-\mathbf{I} \end{aligned} \tag{5}$$

where f_+ (>0) and f_- (<0) are the ultimate tension and compression strengths, respectively.

2.1. *Positive fields*

For a positive definite field, $\mathbf{S}_+$, the stress to be carried by the reinforcement along the principal vectors is

$$\Lambda_u = \Lambda(\mathbf{S}_+) - \boldsymbol{\phi}_+ \tag{6}$$

which can be transformed back to the reference axes through

$$\sigma_u^+ = \mathbf{Q}^{\mathrm{T}}\Lambda_u\mathbf{Q} = \mathbf{S}_+ - f_+\mathbf{I} \tag{7}$$

since $\boldsymbol{\phi}_+$ is hydrostatic. It is assumed that the reinforcement only carries stress along its axis and that, initially, there are three orthogonal bands parallel with the references axes. Thus the plastic resistance tensor provided by the steel is

$$\boldsymbol{\sigma}_{\mathrm{p}} = \mathrm{diag}\,(\sigma_{\mathrm{p}x}, \sigma_{\mathrm{p}y}, \sigma_{\mathrm{p}z}). \tag{8}$$

Finally, the excess capacity of the reinforced material at ultimate is given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathrm{p}} - \boldsymbol{\sigma}_{\mathrm{a}}^{+} = \boldsymbol{\sigma}_{\mathrm{p}} - \mathbf{S}_{+} + f_{+}\mathbf{I} \tag{9}$$

where $\boldsymbol{\sigma}$ is defined by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}. \tag{10}$$

For a safe design, it is clear that σ_x, σ_y and σ_z should be non-negative on every plane, or thus, that $\boldsymbol{\sigma}$ must be positive semi-definite. In addition, there is the obvious practical constraint that $\sigma_{\mathrm{p}x}$, $\sigma_{\mathrm{p}y}$ and $\sigma_{\mathrm{p}z}$ should also be non-negative, so that $\boldsymbol{\sigma}_{\mathrm{p}}$ should also be at least positive semi-definite. Given these constraints, the problem is to minimize the excess strength capacity, and hence the volume of reinforcement. This is best achieved by minimizing the sum of the eigenvalues, τ_i, of $\boldsymbol{\sigma}$. Thus the function to be minimized over $\boldsymbol{\sigma}_{\mathrm{p}}$ is

$$f(\mathbf{y}) = \mathrm{Tr}\,(\boldsymbol{\sigma}) = (\sigma_x + \sigma_y + \sigma_z) \tag{11}$$

where

$$\mathbf{y} = \langle \sigma_{\mathrm{p}x}, \sigma_{\mathrm{p}y}, \sigma_{\mathrm{p}z} \rangle^{\mathrm{t}}. \tag{12}$$

The non-negative constraint on $\boldsymbol{\sigma}$ can be written more succinctly by considering the second and third stress invariants. Baker (1989) deduced that if the second stress invariant, $I_2(\sigma)$, were zero, the system would be over determined in the unknowns $\mathbf{y}$. Hence, at most one eigenvalue should be zero and I_2 positive. The first constraint of $f(\mathbf{y})$ can thus be stated as

$$g_1(\mathbf{y}) = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 > 0. \tag{13}$$

The second constraint comes from the condition that det $(\boldsymbol{\sigma}) \geqslant 0$ [the third invariant $I_3(\boldsymbol{\sigma})$] for positive semi-definiteness, so that

$$g_2(\mathbf{y}) = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \geqq 0. \tag{14}$$

Thus the optimization problem can be written as

$$\min_{\mathbf{y}} f(\mathbf{y})$$

subject to

$$g_1(\mathbf{y}) > 0$$

$$g_1(\mathbf{y}) \geqslant 0 \tag{P_+}$$

and

$$\sigma_{\mathrm{p}x} \geqslant 0, \quad \sigma_{\mathrm{p}y} \geqslant 0, \quad \sigma_{\mathrm{p}z} \geqslant 0.$$

There are a number of ways of solving the program (P_+) for the optimal solution $\mathbf{y}^*$, although for such a small order problem, it is reasonable to examine the Kuhn–Tucker conditions and solve the resulting equations directly.

First, the Lagrangian is written as

$$F_+(\mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{y}) + \lambda_1 g_1(\mathbf{y}) + \lambda_2 g_2(\mathbf{y}) \tag{15}$$

with

$$\boldsymbol{\lambda} = \langle \lambda_1, \lambda_2 \rangle^T. \tag{16}$$

For this problem, the first and second Kuhn–Tucker conditions [see Walsh (1977)] yield:

$$\begin{aligned} &\sigma^*_{px} \geqslant 0, \quad 1 + \lambda_1(\sigma_y + \sigma_z) + \lambda_2(\sigma_y\sigma_z - \tau^2_{yz}) \gtreqless 0 \\ &\sigma^*_{py} \geqslant 0, \quad 1 + \lambda_1(\sigma_x + \sigma_z) + \lambda_2(\sigma_x\sigma_z - \tau^2_{xz}) \gtreqless 0 \\ &\sigma^*_{pz} \geqslant 0, \quad 1 + \lambda_1(\sigma_x + \sigma_y) + \lambda_2(\sigma_x\sigma_y - \tau^2_{xy}) \gtreqless 0 \end{aligned} \tag{17}$$

and the third and fourth conditions yield:

$$\begin{aligned} &\lambda_1^* \geqslant 0, \quad \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_y - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{xz} \gtreqless 0, \\ &\lambda_2^* \geqslant 0, \quad \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} \gtreqless 0. \end{aligned} \tag{18}$$

It should be remembered that for reasons of determinacy, it was deduced that $g_1(\mathbf{y}) > 0$. Hence, from $(18)_1$, it is clear that $\lambda_1^* = 0$. Next, if it is assumed that $\lambda_2^* = 0$, then from (17), the solution would be

$$\sigma^*_{px} = \sigma^*_{py} = \sigma^*_{pz} = 0.$$

However, if this were so, then $\boldsymbol{\sigma}$ would be negative definite (unsafe), unless the applied stress tensor $\boldsymbol{\sigma}_{a+}$ were the trivial null tensor. Hence it can be deduced that $\lambda_2^* > 0$ so that, from $(18)_2$, it must be that

$$g_2(\mathbf{y}) = \det(\boldsymbol{\sigma}) = 0. \tag{19}$$

Given that at most one eigenvalue can be zero, the zero determinant is a clear statement that just one eigenvalue of $\boldsymbol{\sigma}$ must be zero at the optimal point. This fact is the same as that deduced from physical considerations for the 2D slab bending problem: that the excess capacity should be identically zero on one, and only one, principal plane.

Using the revisions to the basic K–T conditions gives a set of non-linear equations in the unknown $\mathbf{y}$ and λ_2 (now written as λ)

$$\mathbf{h}(\mathbf{z}) = \begin{bmatrix} 1 + \lambda(\sigma_y\sigma_z - \tau^2_{yz}) \\ 1 + \lambda(\sigma_x\sigma_z - \tau^2_{xz}) \\ 1 + \lambda(\sigma_x\sigma_y - \tau^2_{xy}) \\ \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} \end{bmatrix} = \mathbf{0} \tag{20}$$

with

$$y_i \geqslant 0, \quad \lambda(=\lambda_2) > 0 \tag{21}$$

and

$$\mathbf{z} = \langle \mathbf{y}^T, \lambda \rangle^T. \tag{22}$$

The solution of (20) can be obtained through Newton iteration

$$\mathbf{z}^{(k+1)} = \mathbf{z}^{(k)} - \mathbf{H}^{-1}(\mathbf{z}^{(k)})\mathbf{h}(\mathbf{z}^{(k)}) \tag{23}$$

after some initial choice $\mathbf{z}^{(0)}$. Here, the derivative of $\mathbf{h}(\mathbf{z})$, $\nabla_z\mathbf{h}(\mathbf{z})$, is the Hessian

$$\mathbf{H}(\mathbf{z}) = \begin{bmatrix} 0 & \lambda\sigma_z & \lambda\sigma_y & (\sigma_y\sigma_z - \tau_{yz}^2) \\ \lambda\sigma_z & 0 & \lambda\sigma_x & (\sigma_x\sigma_z - \tau_{xz}^2) \\ \lambda\sigma_y & \lambda\sigma_x & 0 & (\sigma_x\sigma_y - \tau_{xy}^2) \\ (\sigma_y\sigma_z - \tau_{yz}^2) & (\sigma_x\sigma_z - \tau_{xz}^2) & (\sigma_x\sigma_y - \tau_{xy}^2) & 0 \end{bmatrix}. \tag{24}$$

The formation and inversion of **H** in each iteration is not computationally inefficient with such small matrix orders.

Finally, it should be noted that the solution of $\mathbf{h}(\mathbf{z}) = \mathbf{0}$ may produce several feasible solutions, since the K–T conditions are only necessary conditions. One more condition, the constraint qualification (Walsh, 1977), is required to determine that the solution will be a global minimum. It is easily shown (Baker, 1989) that the Hessian of F_+ is positive semi-definite for all safe solutions so that this final condition is satisfied.

2.2. *Negative fields*

For a negative definite field $\mathbf{S}_-$, the stress to be carried by the steel along the principal directions is

$$\Lambda_a = \Lambda(\mathbf{S}) - \phi \tag{25}$$

which, when transformed to the reference axes, becomes

$$\boldsymbol{\sigma}_a^- = \mathbf{S}_- - \boldsymbol{\phi}_- . \tag{26}$$

Reinforcement is provided in bands parallel with the reference axes, to carry this stress beyond the material failure surface. Here, the reinforcement is to resist a compressive stress, and so the plastic resistance tensor is written as

$$\boldsymbol{\sigma}_p^- = \mathrm{diag}\,(-\sigma_{px}, -\sigma_{py}, -\sigma_{pz}) \tag{27}$$

where $\sigma_{px}, \sigma_{py}, \sigma_{pz}$ are again non-negative.

The tensor of excess capacity is thus given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_p^- - \boldsymbol{\sigma}_a^- = \boldsymbol{\sigma}_p^- - \mathbf{S}_- + f_-\mathbf{I}. \tag{28}$$

Clearly, for a safe solution, the direct stresses in $\boldsymbol{\sigma}$ should be non-positive, with $\boldsymbol{\sigma}$ negative semi-definite. Thus the problem of minimizing the excess capacity can best be achieved by maximizing Tr $(\boldsymbol{\sigma})$. However, to use the same program as for the positive fields, the sign of $\boldsymbol{\sigma}$ is reversed. Define the new tensor as

$$\boldsymbol{\sigma}^- = -\boldsymbol{\sigma} = \boldsymbol{\sigma}_p^- + \mathbf{S} - f_-\mathbf{I}, \tag{29}$$

where the elements of $\boldsymbol{\sigma}^-$ are defined in (10). Here $\boldsymbol{\sigma}^-$ must be, at least, positive semi-definite with non-negativity constraints applied to $\sigma_x, \sigma_y, \sigma_z$ and $\sigma_{px}, \sigma_{py}, \sigma_{pz}$. The maximum has now been transformed to the minimization problem since

$$|\max_{y} \mathrm{Tr}\,(\boldsymbol{\sigma})| = \min_{y} \mathrm{Tr}\,(\boldsymbol{\sigma}^-) \tag{30}$$

where $\mathbf{y} = \langle\sigma_{px}, \sigma_{py}, \sigma_{pz}\rangle^T$ as before. Thus with the definition (29) and

$$f(\mathbf{y}) = \mathrm{Tr}\,(\boldsymbol{\sigma}^-),$$

the program (P_+) can be solved for the solution in a negative field.

2.3. *Mixed fields*

When the applied stress S is indefinite, some parts of the field are positive and some negative so that reinforcement may be required in both tension and compression. Here, strictly speaking, the problem is to minimize the function

$$f(\mathbf{y}) = |\tau_1| + |\tau_2| + |\tau_3|. \tag{31}$$

A simpler solution can be obtained by repeating the program (P_+) for the positive and negative stresses, thus determining the reinforcement required by the positive and negative parts of the field.

(i) Consider first the positive part of the field. Here the applied stress to be used is

$$\mathbf{S}_+ = \mathbf{S}. \tag{32}$$

In solving the program (P_+) using (32), either one or two of the reinforcement stresses will be negative. This must be, since the tensor S is indefinite. It should be noted that a negative steel stress does not yield the required resistance to the negative field. It is physically absurd since even the negative field problem was formulated with the σ_{p_i} non-negative. Moreover, this represents a violation of the mathematical formulation and the appropriate action in such cases is quite clear; no physical reasoning is required as in Wood (1968). Consider separately the cases where the solution to (20) yields one, and two, negative σ_{p_i}.

Firstly, assume for convenience that $\mathbf{h}(\mathbf{z}) = \mathbf{0}$ yields $\sigma^*_{p_z} < 0$. This is a direct violation of the first Kuhn–Tucker condition in $(17)_3$. Hence, the only option is to adopt the second K–T condition, set

$$\sigma^*_{p_z} = 0 \quad \text{and require that} \quad \frac{\partial F_+}{\partial \sigma_{p_z}} > 0. \tag{33}$$

Thus, there are now only three unknowns

$$\mathbf{z} = \langle \sigma_{p_x}, \sigma_{p_y}, \lambda \rangle^{\mathrm{T}} \tag{34}$$

and the set of equations $\mathbf{h}(\mathbf{z})$ becomes

$$\mathbf{h}(\mathbf{z}) = \begin{bmatrix} 1 + \lambda(\sigma_y \sigma_z - \tau_{yz}^2) \\ 1 + \lambda(\sigma_x \sigma_z - \tau_{xz}^2) \\ \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_y\tau_{xy}^2 \end{bmatrix} = \mathbf{0} \tag{35}$$

where $\sigma_z = (-s_{zz} + f_t)$ is known. These equations can be solved algebraically to yield:

$$\sigma_y^* = \frac{\tau_{yz}^2}{\sigma_z} \pm \frac{1}{\sigma_z}\left[\tau_{xz}^2\tau_{yz}^2 + \sigma_z^2\tau_{xy}^2 - 2\sigma_z\tau_{xy}\tau_{yz}\tau_{xz}\right]^{1/2}$$

$$\sigma_x^* = \frac{\tau_{xz}^2}{\sigma_z} \pm \frac{1}{\sigma_z}\left[\tau_{xz}^2\tau_{yz}^2 + \sigma_z^2\tau_{xy}^2 - 2\sigma_z\tau_{xy}\tau_{yz}\tau_{xz}\right]^{1/2}. \tag{36}$$

Obviously the positive root must be chosen. Finally, the σ_{p_i} are found from

$$\begin{aligned} \sigma^*_{p_x} &= \sigma_x^* + [\boldsymbol{\sigma}_s^+]_x \\ \sigma^*_{p_y} &= \sigma_y^* + [\boldsymbol{\sigma}_s^+]_y \\ \sigma^*_{p_z} &= 0. \end{aligned} \tag{37}$$

Should either σ_{px} or σ_{py} be zero, then the other stresses can be found by rotating symbols in (36) and (37). Finally, from either h_1 or h_2,

$$\lambda^* = \frac{-1}{\sigma_y \sigma_z - \tau_{yz}^2}. \tag{38}$$

The requirement that $[1 + \lambda^*(\sigma_x^* \sigma_y^* - \tau_{xy}^{*2})] > 0$ should then be checked.

For the second case assume, again for convenience, that the solution of (20) yields $\sigma_{py}^* < 0$ and $\sigma_{pz}^* < 0$. This is again a violation of the K–T conditions and the necessary alternative option is to set

$$\begin{aligned} \sigma_{py}^* &= 0 \quad \text{with} \quad 1 + \lambda(\sigma_x \sigma_z - \tau_{xz}^2) > 0 \\ \sigma_{pz}^* &= 0 \quad \text{with} \quad 1 + \lambda(\sigma_x \sigma_y - \tau_{xy}^2) > 0 \end{aligned} \tag{39}$$

so that

$$\begin{aligned} \sigma_y &= -s_{yy} + f_t \\ \sigma_z &= -s_{zz} + f_t. \end{aligned} \tag{40}$$

Hence the first and last equations in (20) can be solved directly for λ^* and σ_x^*, respectively,

$$\lambda^* = \frac{-1}{\sigma_y \sigma_z - \tau_{yz}^2} \tag{41}$$

$$\sigma_x^* = \frac{\sigma_y \tau_{xz}^2 + \sigma_z \tau_{xy}^2 - 2\tau_{xy}\tau_{yz}\tau_{xz}}{\sigma_y \sigma_z - \tau_{yz}^2}. \tag{42}$$

Finally, then, the steel stresses for a minimum trace are

$$\sigma_{px}^* = \sigma_x^* + [\sigma_a^*]_x; \quad \sigma_{py}^* = 0; \quad \sigma_{pz}^* = 0. \tag{43}$$

(ii) Having designed the reinforcement for the tension field, attention is turned to the negative field. The applied stress is thus

$$S_{.} = S \tag{44}$$

which is used to find the compression reinforcement as shown for the negative fields. The important point to remember is that the program in a negative field is rewritten by reversing the sign so that the problem is to find the minimum positive reinforcement in a positive field. Hence the previous algorithm is repeated using expression (29) with (44), where σ_a^- is now indefinite. Thus, as before, at least one σ_{pi}^* in the solution of (20) will be negative, and so the formulae (36), (37) or (42), (43) should be used for the final design.

Thus part (i) using S positively reinforces the positive domain of the stress space, and part (ii) using S positively reinforces the negative domain of the stress space; illustrative examples of the solution process are given in Baker (1989).

3. MOMENT FIELDS IN SKEW SLABS

The formulation of the two-dimensional problem follows the previous section exactly. However, rather than repeating the stress problem, the problem of determining the optimum plastic moment of resistance in a skew slab will be considered. That is, the two bands of reinforcement are not orthogonal, but meet at some angle α; for convenience one band is

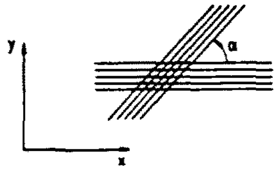

Fig. 1. Skew reinforcement.

assumed to be parallel with the x axis (Fig. 1). It is also assumed that the concrete carries no tension, so that the plastic resistance is provided entirely by the moments of the reinforcement nets. Denote the plastic moments provided by the reinforcement as m_{px} and $m_{p\alpha}$. Since the moment vectors m_{px} and $m_{p\alpha}$ are not orthogonal, the moment of resistance tensor, $\mathbf{M}_p$, is not diagonal. Its elements can be simply evaluated by rotating a separate tensor for the "α" system into the reference axis system, and adding m_{px}:

$$\mathbf{M}_p = \begin{bmatrix} m_{px}+m_{p\alpha}\cos^2\alpha & -m_{p\alpha}\sin\alpha\cdot\cos\alpha \\ -m_{p\alpha}\sin\alpha\cos\alpha & m_{p\alpha}\sin^2\alpha \end{bmatrix}. \tag{45}$$

Since it is assumed that the slab has no inherent resistance moment without reinforcement,

$$\phi_+ = \phi_- = 0 \tag{46}$$

and the applied moment tensor is, therefore,

$$\mathbf{M}_a = \begin{bmatrix} m_{ax} & m_{axy} \\ m_{axy} & m_{ay} \end{bmatrix}. \tag{47}$$

3.1. *Positive fields*

For a positive field

$$\mathbf{M}_a^+ = \mathbf{M}_a \tag{48}$$

with det $(\mathbf{M}_a) > 0$; $m_{ax}, m_{ay} > 0$. The excess moment resistance tensor is thus

$$\mathbf{M} = \mathbf{M}_p - \mathbf{M}_a \tag{49}$$

which will be written as

$$\mathbf{M} = \begin{bmatrix} m_x & m_{xy} \\ m_{xy} & m_y \end{bmatrix} \tag{50}$$

where

$$\begin{aligned} m_x &= m_{px}+m_{p\alpha}\cos^2\alpha - m_{ax} \\ m_y &= m_{p\alpha}\sin^2\alpha - m_{ay} \\ m_{xy} &= -m_{p\alpha}\sin\alpha\cos\alpha - m_{axy}. \end{aligned} \tag{51}$$

Following the same reasoning as before, the optimization problem, which seeks a solution for the unknowns

$$\mathbf{y} = \langle m_{px}, m_{p\alpha}\rangle^T, \tag{52}$$

becomes

$$\min_{y} [f(y) = \mathrm{Tr}(M)]$$

subject to

$$g(\mathbf{y}) = \det(M) = 0 \qquad (P_{2+})$$

and

$$m^*_{px} \geq 0, \quad m^*_{pz} \geq 0.$$

After writing the Lagrangian

$$F(\mathbf{y}, \lambda) = f(\mathbf{y}) + \lambda g(\mathbf{y}) \tag{53}$$

the Kuhn–Tucker conditions yield the equations

$$\mathbf{h}(\mathbf{z}) = \begin{bmatrix} 1+\lambda m_y \\ 1+\lambda(m_y \cos^2\alpha + m_x \sin^2\alpha + 2m_{xy}\cos\alpha\sin\alpha) \\ m_x m_y - m_{xy}^2 \end{bmatrix} = \mathbf{0}. \tag{54}$$

Here, the system can be solved algebraically. Clearly, from the first two equations in (54),

$$m_y = m_x + 2m_{xy}\cot\alpha \tag{55}$$

which when substituted in $(54)_3$ yields the solution (Baker, 1989),

$$m_y = \frac{m_{xy}}{\sin\alpha}(\cos\alpha + \mathrm{sgn}(m_{xy})) \tag{56}$$

in the range of skew $0 < \alpha < \pi$; physically the second half plane $\pi < \alpha < 2\pi$ is the same. Using (56) in $(51)_{2,3}$ gives the plastic moment

$$m_{p\alpha} = \frac{m_{ay}}{\sin^2\alpha} + \left[\frac{m_{axy} + m_{ay}\cot\alpha}{\sin\alpha}\right]. \tag{57}$$

Next, substituting (57) in (55) yields

$$m_x = \frac{m_{xy}}{\sin\alpha}(\mathrm{sgn}(m_{xy}) - \cos\alpha) \tag{58}$$

which must be non-negative. Finally from (51), (57) and (58), we find

$$m_{px} = m_{ax} + 2m_{axy}\cot\alpha + m_{ay}\cot^2\alpha + \left[\frac{m_{axy} + m_{ay}\cot\alpha}{\sin\alpha}\right] \tag{59}$$

which can also be guaranteed non-negative (Baker, 1989).

3.2. *Negative fields*

Here

$$\mathbf{M}_a^- = \mathbf{M}_a \tag{60}$$

with $m_{ax}m_{ay} - m_{axy}^2 > 0$; $m_{ax}, m_{ay} < 0$. The excess moment tensor, is given by

$$\mathbf{M} = \mathbf{M}_p^- - \mathbf{M}_a^- \tag{61}$$

where the plastic resistance tension $\mathbf{M}_a^-$ is exactly like (45) except that the moments are written with a superscript and m_{px}^-, $m_{p\alpha}^-$ must be non-positive. It should be noted that the sign of the plastic moments is dictated by the position of the net (top or bottom of the slab), but that the stress in the steel is always positive; in the stress problem, a negative σ_p implied a compressive stress. Hence, the problem could be rewritten as a minimization problem with non-negative m_p values. Alternatively, the non-positive $\mathbf{M}_p^-$ can be retained, and a maximum of Tr ($\mathbf{M}$) be found.

Either approach leads to a set (54). Here the solution which guarantees a non-positive m_x and m_y is taken. The result is

$$m_{px}^- = m_{ax} + 2m_{axy}\cot\alpha + m_{ay}\cot^2\alpha - \left[\frac{m_{axy} + m_{ay}\cot\alpha}{\sin\alpha}\right] \tag{62}$$

$$m_{p\alpha}^- = \frac{m_{ay}}{\sin^2\alpha} - \left[\frac{m_{axy} + m_{ay}\cot\alpha}{\sin\alpha}\right] \tag{63}$$

which are always non-positive.

3.3. *Mixed fields*

In a mixed field, $\mathbf{M}_a$ is indefinite with det ($\mathbf{M}_a$) < 0

$$m_{ax}m_{ay} - m_{axy}^2 < 0. \tag{64}$$

Thus the reinforcement will be required in one, or probably two, directions (x, α) in both the top and bottom of the slab. Here, the minimum and maximum trace conditions should be replaced by a minimization of the sum of the absolute values of the principal excess moments, for both top and bottom of the slab, $\mathbf{M}^+$ and $\mathbf{M}^-$. However, it is preferable to repeat the minimization process for both top and bottom reinforcement, from the previous sections, since a minimization in the form of expression (31) can only require a greater reinforcement volume. Now, of course, the non-negativity of (57) and (59), or the non-positivity of (62) and (63) cannot be guaranteed *a priori*, though these expressions can be rewritten as bounds to m_{ax} and m_{ay}, and are in fact the boundaries to Morley's domains. Should any element of the solution contravene the conditions

$$m_{px}, m_{p\alpha} > 0; \quad m_{px}^-, m_{p\alpha}^- < 0 \tag{65}$$

then it is a requirement of the Kuhn–Tucker conditions that the offending element be set to zero, and the corresponding equation in $\mathbf{h}(\mathbf{z})$ becomes an inequality. Physically, this is the same as saying that it is nonsensical to provide top reinforcement to resist a positive moment; the fact that m_{px} and $m_{p\alpha}$, say, may be of opposite sign is just a result of the mathematical minimization.

Once one unknown is zero, the second can be obtained directly from the $g(y) = 0$ condition. Thus, if $m_{p\alpha} < 0$ from (57), then $m_{p\alpha}$ is set to zero and m_{px} is found from det ($\mathbf{M}$) = 0, giving

$$m_{px} = m_{ax} + \frac{m_{axy}^2}{|m_{ay}|}$$

$$m_{p\alpha} = 0. \tag{66}$$

Similarly, if $m_{p\alpha}^- > 0$, m_{px}^- can be recalculated

$$m_{py} = m_{ay} - \frac{m_{axy}^2}{|m_{ax}|}. \tag{67}$$

From (64) it can be shown that m_{py} will always be positive, and m_{py}^- negative, as required. Expressions (66) and (67) agree with those for an orthogonal set, but differ from those in Armer (1968) which contain functions of α. However, if either $m_{p\alpha}$ or $m_{p\alpha}^-$ does not exist, then an angle α for the corresponding m_{py} or m_{py}^- has no meaning. It is clear that the constraints on $m_{p\alpha}$ and $m_{p\alpha}^-$ cannot both be contravened for a given $\mathbf{M}_a$ so that two bands of reinforcement will be needed at one surface, and one at the other.

If either m_{py} or m_{py}^- contravened the non-negative or non-positive conditions respectively, the recalculated plastic moments are

$$\begin{aligned} m_{py}, m_{py}^- &= 0 \\ m_{p\alpha}, m_{p\alpha}^- &= \frac{1}{\sin^2\alpha}\left[\frac{m_{ax}m_{ay} - m_{axy}^2}{m_{ax} + 2m_{axy}\cot\alpha + m_{ay}\cot^2\alpha}\right]. \end{aligned} \tag{68}$$

Again it can be shown that both constraints cannot be contravened for a given $\mathbf{M}_a$.

4. FAILURE SURFACES—YIELD LINES

The condition $g(\mathbf{y}) = 0$ which was deduced through the K–T condition in Section 2.1 could be reasoned on physical grounds since it corresponds to one principal excess stress/moment being zero. That is, the minimum is found when the excess capacity is zero on some plane. This condition is thus a yield condition and the statements

$$\begin{aligned} \det(\boldsymbol{\sigma}) &= 0 \\ \det(\mathbf{M}) &= 0 \end{aligned} \tag{69}$$

are just yield criteria for any of the formulations considered.

Moreover, it is clear from the similarity transform (2), and the yield conditions (69), that the eigenvectors of the excess capacity tensor, either $\boldsymbol{\sigma}$ for stress problems or $\mathbf{M}$ for bending in slabs, are the direction cosine vectors of the failure surface. Thus, one eigenvector $\mathbf{q}_2$ corresponding to the yield condition (zero eigenvalue) is the normal to the yield surface and, by the normality rule, it is tangential to the yield line at that point. From the orthogonality of eigenvectors, the second eigenvector, $\mathbf{q}_1$, must be tangential to the yield surface and orthogonal with the yield line.

To obtain an expression for the failure angles, θ_1 and θ_2, it is only necessary to expand, say, $\mathbf{M}^*\mathbf{q} = \mathbf{0}$. After eliminating the moments using (56) and (51) for a positive field, we obtain:

$$\tan\theta_1 = -\frac{(1 \pm \cos\alpha)}{\sin\alpha} = -\cot\frac{\alpha}{2} \quad \text{or} \quad -\tan\frac{\alpha}{2}. \tag{70}$$

Thus, the failure angle is

$$\theta_f = \frac{\alpha}{2} \quad \text{or} \quad \frac{\alpha}{2} \pm \frac{\pi}{2}. \tag{71}$$

In other words, when the optimum reinforcement is used, the yield line bisects the angle between the reinforcement bands. The value to be used from (71) depends on the relative magnitude of m_x and m_y

$$m_x > m_y \quad \theta_f = \frac{\alpha}{2} \pm \frac{\pi}{2}$$

$$m_x < m_y \quad \theta_f = \frac{\alpha}{2}. \tag{72}$$

Note that (72) applies whether or not m_{px} and $m_{p\alpha}$ are equal.

The same result can be shown to apply in the case of negative fields, as would be expected. For a mixed field, where either m_{px} or $m_{p\alpha}$ is zero, the eigenproblem can be solved, but no important result like (72) is generally applicable.

5. OPTIMUM SKEW

In practice, orthogonal nets are most common, and if a skew is adopted, it is usually to suit some geometric constraint, such as bridge skew. If, however, skew can be chosen, then the design can be made even more economical.

5.1. *Moments in slabs*

Using condition (71), Morley (1969) reasoned that the optimum design occurred when the skew, α, was twice the principal angle of the applied moment triad so that failure would occur along the principal plane. That is,

$$\tan\alpha = \frac{-2\cdot m_{axy}}{m_{ax} - m_{ay}}. \tag{73}$$

It is true that this condition yields $m_{px} = m_{p\alpha}$, but it does not yield the absolute minimum moment volume, V_m. For a positive field, it is a requirement that m_x, m_y be non-negative, and thus the absolute allowable minimum is given by

$$\mathrm{Tr}\,(\mathbf{M}) = 0 \tag{74}$$

corresponding to

$$V_m = (m_{px} + m_{p\alpha}) = (m_{ax} + m_{ay}) \tag{75}$$

which follows when the absolute term in (57) and (59) is zero. That is, the optimum skew corresponds to

$$\tan\alpha = -\frac{m_{ay}}{m_{axy}}. \tag{76}$$

Moreover, the skew (76) when substituted into (57) and (59) gives

$$m_{px} = m_{ax} - \frac{m_{axy}^2}{m_{ay}}$$

$$m_{p\alpha} = m_{ay} + \frac{m_{axy}^2}{m_{ay}} \tag{77}$$

which are guaranteed positive definite by the definition of a positive field. It is interesting to note that the optimum resistance moments, given the optimum skew, are actually independent of that skew.

For a negative field, the same condition (75) can be applied, and this gives rise to identical expressions to (77); here m_{px}^- and $m_{p\alpha}^-$ are negative definite by the definition of a negative field.

In a mixed field, the optimum skew depends on whether one or two bands of reinforcement are required. For the bottom steel, if the skew is optimized, the reinforcement requirements are given by (77) with skew (76). However, either $m_{p\alpha}$ or $m_{p\alpha}$ in (77) must be negative. If $m_{ay} < 0$, then $m_{p\alpha}$ would be negative so that expressions (66) should be used and skew is irrelevant.

If $m_{ay} > 0$, then m_{px} would be negative. However, expressions (68) with skew (76) will not yield the minimum volume since (76) assumes both moments positive (Baker, 1989). Nonetheless, it is intuitively obvious that with one negative principal moment, the optimum skew would correspond to the principal angle with $m_{p\alpha}$ equal to the first principal moment of $\mathbf{M}_a$.

5.2. *Stress problem*

For the general 3D stress problem, no effort is made to study the mechanics of behaviour through an algebraic solution. Here, the non-linear program will be reformulated for non-orthogonal reinforcement bands, leaving the numerical solution to given cases.

To find the resistance tensor, σ_p, plastic stress σ_{pi} are considered for each band defined by the non-orthogonal axes α_1, α_2 and α_3. Following the approach in Section 3, the individual resistance tensors are null except that $(\sigma_{pi})_{11} = \sigma_{pi}$. The tensor $\boldsymbol{\sigma}_p$ is thus found by rotating the three $\boldsymbol{\sigma}_{pi}$ into the reference set (x, y, x) using the 3D rotation matrices of direction cosines. For convenience, axis x will be defined as parallel with α_1. Thus, the resistance tensor can be written as

$$\sigma_{pij} = \sigma_{p1}(\delta_{i1}\delta_{1j}) + \sigma_{p2}(l_{1i}l_{1j}) + \sigma_{p3}(n_{1i}n_{1j}) \tag{78}$$

where $i, j\,(=1, 2, 3)$ refer to the orthogonal axes. We note that the α_2 and α_3 axes themselves belong to orthogonal sets but that the direction of the second two axes, and also the cosines l_{ij} and n_{ij}, $i > 1$, do not enter the formulation.

In this section, only the formulation for positive and mixed fields will be given since the extension follows as before. Thus, writing the components of $\boldsymbol{\sigma}_a^+$ as σ_{aij} the elements of the excess tensor are now

$$\sigma_{ij} = \sigma_{p1}(\delta_{i1}, \delta_{1j}) + (\sigma_{p2}(l_{1i}l_{1j}) + \sigma_{p3}(n_{1i}, n_{1j}) - \sigma_{aij}. \tag{79}$$

Here, the direction cosines are unknowns, so that

$$\mathbf{y} = \langle \sigma_{p1}, \sigma_{p2}, \sigma_{p3}, l_{11}, l_{12}, l_{13}, n_{11}, n_{12}, n_{13} \rangle^t. \tag{80}$$

The conclusions made regarding the second and third invariants of $\boldsymbol{\sigma}$ still apply, and this the optimization problem becomes

$$\min_{\mathbf{y}} f(\mathbf{y})$$

subject to

$$\begin{aligned} g_1(\mathbf{y}) &> 0 \\ g_2(\mathbf{y}) &= 0 \\ g_3(\mathbf{L}) &= 0 \qquad\qquad (P_m) \\ g_4(\mathbf{N}) &= 0 \end{aligned} \tag{81}$$

with

$$\sigma_{pi} \geqslant 0 \quad i = 1, 2, 3$$

where

$$f(\mathbf{y}) = \mathrm{Tr}\,(\boldsymbol{\sigma}) = \sum_{i=1}^{3} (\sigma_{pi} - \sigma_{aii}) \tag{82}$$

(where no summation is implied by ii), and

$$\begin{aligned} g_2(\mathbf{y}) &= \det(\boldsymbol{\sigma}) \\ g_3(\mathbf{L}) &= l_{11}^2 + l_{12}^2 + l_{13}^2 - 1 \\ g_4(\mathbf{N}) &= n_1^2 + n_{12}^2 + n_{13}^2 - 1. \end{aligned} \tag{83}$$

The Lagrangian is thus

$$F_+(\mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{y}) + \lambda_2 g_2(\mathbf{y}) + \lambda_3 g_3(\mathbf{L}) + \lambda_4 g_4(\mathbf{N}) \tag{84}$$

where $\boldsymbol{\lambda} = \langle \lambda_2, \lambda_3, \lambda_4 \rangle^T$ and $\lambda_1 = 0$. After differentiating $F_+(\mathbf{y}, \boldsymbol{\lambda})$, the K–T conditions yield a set of non-linear equations in the unknowns $\mathbf{z} = \langle \mathbf{y}^T, \lambda^T \rangle^T$,

$$\mathbf{h}(\mathbf{z}) = \mathbf{0}. \tag{85}$$

The explicit form of $\mathbf{h}(\mathbf{z})$ is given in Baker (1989). These equations can be solved in a number of ways. If a Newton solution were adopted, the Hessian $\mathbf{H}(\mathbf{z})$ would be symmetric though not all diagonals would be zero here. It should also be noted that the cost of a Newton solution of this 12×12 system could be significant.

As before, should any of the non-negativity conditions be violated in a mixed field, the offending σ_{pi} is set to zero, removed from $\mathbf{y}$ (and $\mathbf{z}$) and the relevant equations in $\mathbf{h}(\mathbf{z})$ removed. For the plane stress problem, the program can be reduced to just four equations, with the unknown skew represented by α:

$$\mathbf{z} = \langle \sigma_{pi}, \sigma_{p2}, \alpha, \lambda \rangle^T \tag{86}$$

$$\mathbf{h}(\mathbf{z}) = \begin{bmatrix} 1 + \lambda\sigma_{22} \\ 1 + \lambda(\sigma_{22}\cos^2\alpha + \sigma_{11}\sin^2\alpha - \sigma_{12}\cos\alpha\sin\alpha) \\ 2\lambda\sigma_{p2}(\sigma_{22}\cos\alpha - \sigma_{12}\sin\alpha) \\ \sigma_{11}\sigma_{12} - \sigma_{12}^2 \end{bmatrix}. \tag{87}$$

There is no need to solve the program since the deductions made for the moment field problem apply here to the two dimensional stress problem.

6. NUMERICAL EXAMPLES

6.1. *Compressed panel*

The programs for the two-dimensional stress problem will now be used to demonstrate certain features of optimum reinforcement requirements in the end block of a compressed panel. Figure 2 shows the finite element mesh used to obtain the stress levels. Failure

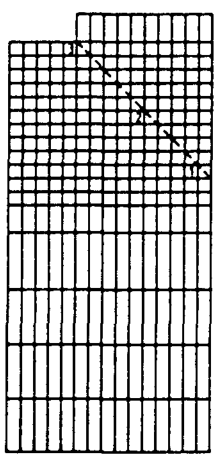

Fig. 2. Mesh layout for compressed panel.

Table 1. Optimum reinforcement for compressed panel. Stresses were evaluated for a load of 200 kN. Material constants $f_t = 0$ N m^{-2}, $f_c' = -20$ N mm^{-2}

	Element		1	2	3
FE stresses		σ_x	2.472	1.5	−13.348
		σ_y	−21.774	−22.088	−17.627
		τ_{xy}	−0.656	−2.362	−13.608
		σ_1	2.492	1.736	−1.715
		σ_2	−21.806	−22.341	−29.326
		θ_p	−1.5	−5.7	−40.5
Optimum reinforcement	Orthogonal (x, y)	σ_{px}	2.492	1.752	0
		σ_{py}	1.794	2.348	11.236
		V	4.286	4.100	11.236
	Orthogonal θ_p	σ_{px}	2.492	1.736	0
		σ_{py}	1.806	2.342	9.326
		V	4.298	4.078	9.326
	$\alpha = 45$	σ_{px}	2.492	1.752	0
		σ_{py}	4.526	12.024	11.144
		V	7.018	13.776	11.144
	$\alpha = 135^\circ$	σ_{px}	2.492	1.752	0
		σ_{py}	3.946	4.182	9.348
		V	6.438	5.934	9.348

occurred [see Baker (1989)] at 200 kN, hence elastic stresses at 100 kN were doubled to give design ultimate values. The tensile strength was again set to zero and $f_c' = -20$ N mm^{-2}. Results are detailed for just three elements lying along the line of the "wedge face" (see Fig. 2); the stresses along this line are of the course the most interesting.

Table 1 lists firstly the stresses and principal stresses followed by the calculated pointwise reinforcement for four reinforcement patterns: an orthogonal net aligned in the (x, y) system, an orthogonal net aligned to the principal stress axes and two skew nets with one band parallel to x and the other angles of 45° and 135°, respectively. The stress fields for elements 1 and 2 were mixed, but for element 3, it was negative definite.

Mathematically, of course, many of the steel stress values would be negative to cope with the negative areas of the fields. The stresses given represent the required areas. Written as resistance stress, the tabulated values provide resistance to the applied stress in excess of the yield surface at any orientation.

It is of course impractical to align reinforcement along principal planes, though it is clear from element 3 that considerable savings would be made were it possible. In fact, if a field is positive or negative, and in many mixed fields, then such an alignment will yield the absolute minimum reinforcement volume; this can be seen by reference to Mohr's circle noting that the reinforcement stresses required are the principal stress of applied stress. In some mixed fields, however, an orthogonal net in the reference axes—or some other—may yield a smaller volume as was the case for element 1. It should be remembered that, being a mixed field, the reinforcement areas σ_{px} and σ_{py} were determined from the positive and negative strategies; for this 2D case, equivalent formulae like (66) and (68) are directly applicable. Comparison of these formulae with Mohr's circle confirms that the observed relationship was quite reasonable.

Although the applied stress tensor in element 3 was negative definite, the tensor in excess of the compression failure surface was mixed. That is,

$$\sigma_\Delta^- = \begin{bmatrix} 6.652 & -13.608 \\ -13.608 & 2.373 \end{bmatrix}$$

with eigenvalues 9.26 and −18.29. Since the stress of 6.652 lies within the surface, and is not actually an applied stress, there is no need to reinforce in both directions, unlike elements 1 and 2 which require steel to resist positive and negative stress.

As regards the choice of skew, the two values, along and orthogonal to the 45° plane, were adopted because cracks form along the shear plane/wedge face. In every case the skew

Table 2. Optimum reinforcement for a compressed cube. Stresses were evaluated for a uniform pressure of $20\ \mathrm{N\ mm^{-2}}$. Material constants $f_t = 0\ \mathrm{N\ mm^{-2}}$, $f_c = -20\ \mathrm{N\ mm^{-2}}$

Position	Node 1	Node 2
FE stresses		
σ_{px}	0.159	−3.306
σ_{py}	0.159	−5.440
σ_{pz}	−22.384	−26.876
τ_{xy}	−0.035	0.242
τ_{yz}	−0.305	1.235
τ_{xz}	−0.305	−2.424
Optimum reinforcing		
σ_{px}	0.199	0
σ_{py}	0.199	0
σ_{pz}	2.393	7.032
Volume	2.782	7.032

$\alpha = 135°$ gave a smaller volume than $\alpha = 45°$. This is because the shear crack actually forms by rupture of many small struts formed by cracks across the wedge face. These minimum volumes, which represent an efficient choice of skew, reflect this fact. Since $\sigma_{px} = 0$ for all nets in element 3, it is clear that the optimum skew would correspond to the principal angle.

6.2. *Compressed cube*

For a second example, the programs for the three-dimensional stress problem will be used to determine reinforcement requirements at selected points in a cubic concrete footing ($1500 \times 1500 \times 1500$ mm). The cube was loaded through a 750 mm square bearing plate carrying a uniform pressure of $20\ \mathrm{N\ mm^{-2}}$. The elastic stresses were found using a finite element mesh of 32 20-node isoparametric bricks, using reduced integration. Nodal stresses were obtained by linearly interpolating from the eight Gauss points to the element faces, and then linearly to the nodes. Only two nodal positions will be examined, for brevity. Considering a 3D representation of Fig. 2, the four 45° wedge faces meet at a single point on the centreline of the cube, denoted node 1. Node 2 lies midway along the edge of the bearing plate.

Table 2 shows the FE stresses at nodes 1 and 2, together with the reinforcement requirements for an orthogonal net at each site; the requirements for the positive and negative stresses have been combined. The same material strengths as in the last example were used. At both positions, some sensitivity was noted with regard to the choice of starting vector for Newton iteration, but there was no difficulty in achieving the results.

The applied stress tensor at node 2 was negative definite, thus requiring no positive reinforcement, whereas the tensor in excess of the compression failure surface was mixed. In fact, the optimum gave both σ_{px} and σ_{py} as negative so that (42) and (43) were used to find the steel requirements in the z direction. At node 1, the stress tensor in excess of both the tensile and compressive failure surfaces was mixed. Reinforcing for positive stress gave a negative σ_{pz}, so that expressions (36) and (37) were used to find the x and y requirements, whereas reinforcing for negative stress again yielded two negative values so that (42) and (43) were used for stress in the z direction.

7. FE SYNTHESIS AND GLOBAL OPTIMALITY

A simplified finite element synthesis might be proposed, whereby the structure was discretized and analyzed under design ultimate loads. Thus the pointwise optimization would be carried out over the structure, which would be reanalyzed, including the new steel stiffness, and the process iterated until some tolerance on reinforcement densities was met.

The main objection to this process is that it may not yield the globally optimal solution since it is comprised of the sum of minima, not the minimum of a sum. That is, the globally optimal solution would require the minimum total reinforcement volume. In the case of a three-dimensional stress problem, the objective function would thus be

$$\Phi(\sigma) = \int_{\Omega} (|\lambda_1(\sigma)| + |\lambda_2(\sigma)| + |\lambda_3(\sigma)|) \, d\Omega \tag{88}$$

where the λ_i are the eigenvalues of the pointwise excess stress tensor $\boldsymbol{\sigma}$; note that (88) is very like Strang and Kohn's (1983) Michell problem. The volume integrated in (88) would be carried out over discrete points (Gauss stations) in a finite element formulation. However, given a strength criterion, in a sense, it does not matter whether pointwise or global statements are used, since local failures are not permitted and the approach reinforces everywhere. This is, of course, unnecessary in reinforced concrete since unreinforced cracked zones at ultimate are quite acceptable. Impractical reinforcement patterns might be rationalized, or simply chosen by practical choice or constraint, though perhaps at the expense of optimality, which expects failure at all points at the same load.

The true optimum will be found from a solution which fuses the optimum layout problem with the continuum stress problem, and which embodies sensitivity analysis, because of the effect of stress flow on optimum reinforcement. For practical purposes, the simple algorithm outlined above should suffice.

8. CONCLUSIONS

A consistent approach to the provision of optimum pointwise reinforcement to resist fracture has been presented. The method was to define an excess plastic capacity tensor and to write the governing non-linear program in terms of the invariants of the tensor. The Kuhn–Tucker optimality conditions were explicitly determined for the two- and three-dimensional cases, including non-orthogonal nets and optimum skew. The resulting equations could be solved numerically using a variety of techniques, since the order is small, and in certain cases they were solved algebraically. The latter gave specific physical insight to the meaning of the resulting formulae in the two-dimensional and plate bending cases: (i) when the skew angle is twice the principal angle of the applied moment triad, the required moments of resistance are equal, (ii) optimum skew can be found from $\tan\alpha = -m_{ay}/m_{axy}$ if two bands of reinforcement are needed, (iii) in a mixed field, where just one band is required, this should be oriented to the principal angle of the moment triad.

The technique presented provides understanding of the mechanics of the problem, both through the solutions obtained and, where solutions were known, through a study of the optimization process. The appeal of the approach lies in being able to deduce directly from the Kuhn–Tucker conditions so many of the physical properties of the problem. Notwithstanding, the technique represents a powerful lower bound approach to the design of fracturing materials such as reinforced concrete.

Acknowledgement—The author wishes to thank one of his research students, Mr C. A. Grummitt, who performed the finite element calculations for the second example.

REFERENCES

Armer, G. S. T. (1968). Discussion of Wood, R. H. (1968). *Concrete* **2**, 319–321.
Baker, G. (1989). On the yield condition and optimum pointwise reinforcement for fracturing continua. Civil Engineering Research Report No. CE102, University of Queensland.
Clark, L. A. (1976). The provision of tension and compression reinforcement to resist in plane forces. *Mag. Conc. Res.* **28**, 3–12.
Morley, C. T. (1969). Skew reinforcement of concrete slabs against bending and torsional moments. *Proc. Inst. Civ. Engrs* **42**, 57–74.
Rozvany, G. I. N. and Wang, C. M. (1984). Optimal layout theory: allowance for self-weight. *J. Engng Mech.* **110**, 66–83.
Strang, G. and Kohn, R. V. (1983). Hencky–Prandtl nets and constrained Michell trusses. *Comp. Meth. Appl. Mech. Engng* **36**, 207–222.
Walsh, G. R. (1977). *Methods of Optimization*. John Wiley, London.
Wood, R. H. (1968). The reinforcement of slabs in accordance with a predetermined field of moments. *Concrete* **2**, 69–76.